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ON THE THEORY OF AGE-DEPENDENT STOCHASTIC BRANCHING PROCESSES

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ON THE THEORY OF AGE-DEPENDENT STOCHASTIC BRANCHING PROCESSES[1]

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§1. Introduction.

~~We are interested in investigating~~ the following problem is investigated which is of possible biological, chemical and physical interest. A particle existing at time $t = 0$ is assumed to have probabilities q_n, $n \geq 0$, of being transformed into n similar particles at some random time $t > 0$. Assume that ~~we~~ a start is made with a single particle at $t = 0$. Under the hypothesis that any particle has a life-length probability distribution independent of its time of birth and of the number of other particles existing at this time, the problem is to determine the probability distribution of $Z(t)$, the number of particles in existence at time t. → to P. 2

The simplest case, and the one most often considered previously, is that where the probability that a particle in existence at t be transformed between t and $t + \Delta t$ is $a\Delta t + o(\Delta t)$ and is thus independent of age and of absolute time. Here the cumulative distribution $G(t)$ of the random transformation times has the form $G(t) = 1 - e^{-at}$. For this particular case, the problem is more tractable due to the convenient fact that the non-linear integral equation which is obtained in the general case reduces to an ordinary non-linear differential equation which in the case of binary splitting is of Bernoulli type and hence can be solved in elementary terms; see, for example, D. G. Kendall, [4].

[1] Expansion of results announced in Proceedings of the National Academy of Sciences, Vol. 34 (1948), pp. 601-604.

Unfortunately, the assumption that the probability of transformation is independent of the age of the particle is not realistic in many cases of interest. Rather, it is more likely that the distribution of transformation times is concentrated about a certain mean life length. This is particularly likely to be true in biological phenomena such as the growth of a colony of bacteria.

In our work, we assume that the random transformation times have a cumulative distribution $G(t)$, where $G(0+) = 0$, $G(\infty) = 1$. Depending upon what we wish to prove, further assumptions are added. The precise restrictions will be given below.

from P.1

The problem is restricted ~~We shall restrict ourselves,~~ as far as detailed exposition goes, to the special case where only binary transformations occur; that is, one particle can be transformed only into two others. This is the most important case biologically, and the methods employed to deal with this case are easily extended~~, as will be pointed~~ out later, to deal with the general case with n-ary transformations.

It should be mentioned that D. G. Kendall [5] has recently treated the case where $G(t)$ is a k-fold convolution of distributions of the form $1 - e^{-at}$. In this case the process can be considered as a Markoff process involving k types of particles.

We now make some definitions and assumptions. Set

$$p_r(t) = \mathrm{Prob}[Z(t) = r], \qquad r = 0, 1, \cdots$$

$$F(s, t) = \sum_{r=0}^{\infty} p_r(t) s^r .$$

The generating function,

$$F(s, t) = \sum_{r=0}^{\infty} p_r(t) s^r \tag{1}$$

where

(2) $$p_r(t) = \mathrm{Prob}[Z(t) = r], \qquad r \geq 0,$$

and $Z(t)$ is, as in the first paragraph, the number of particles in existence at time t, will be the focal point of our investigations.

Setting

(3) $$h(s) = \sum_{n=0}^{\infty} q_n s^n$$

where q_n, as above, is defined to be the probability that a particle is transformed into n particles when transformation occurs, standard probabilistic reasoning leads to the non-linear integral equation

(4) $$F(s, t) = \int_0^t h[F(s, t-y)]dG(y) + s(1 - G(t)) .$$

The classical method of successive approximations yields the result that there is a unique bounded solution of (4) for $|s| \leq 1$, which possesses all the elementary properties of a generating function. We next consider the distribution of the random variable defined by

(5) $$W(t) = Z(t)/E(Z(t)) .$$

To this end, we require the existence and asymptotic behavior of the expectation of $Z(t)$, $E[Z(t)] = m_1(t)$. The expectation satisfies the linear integral equation

(6) $$m_1(t) = b_1 \int_0^t m_1(t-y)dG(y) + 1 - G(t), \qquad (b_1 = \sum_1^{\infty} nq_n) .$$

In this paper, we discuss only the case where $1 < b_1 < \infty$. In this case there is a positive probability that the family of particles will not become extinct.

Equation (6) is a special case of the familiar equation of renewal theory which has been treated generally by Feller [2], Täcklind [7], and Bellman and Harris [1]. In the present case there are special conditions satisfied which enable more precise asymptotic results to be obtained.

Under the assumption that $b_1 > 1$ we show that the random variable $W(t)$ converges in mean square to a random variable w. The connection between $Z(t)$ and w is as follows. Let $K(u)$ be the distribution function of w and set

$$(7) \qquad \phi(s) = \int_{0-}^{\infty} e^{su} dK(u), \qquad Re(s) \leq 0.$$

Then it follows that

$$(8) \qquad \phi(s) = \lim_{t \to \infty} F\left(e^{s/e^{at}}, t\right), \qquad Re(s) \leq 0,$$

and consequently that $\phi(s)$ satisfies the non-linear integral equation

$$(9) \qquad \phi(s) = \int_{0}^{\infty} h[\phi(s/e^{at})] dG(t) .$$

The term e^{at} occurs because of the fact we prove below that $E(Z(t)) \sim m_1 e^{at}$ as $t \to \infty$; the constant $\underline{a}$ will be defined below.

The properties of the solutions of equation (9) are now studied. It is shown that for the solution of (9) of the form (7),

we have

$$(10) \qquad \phi(it) \longrightarrow 0, \quad \text{as } t \longrightarrow \pm\infty,$$

if $h(0) = 0$, where $h(s)$ is defined by (3). A consequence of this is that $K(u)$ is continuous in u, except for a discontinuity at $u = 0$ which occurs only when $h(0) \neq 0$. Imposing a further condition of the type

$$(11) \qquad 1 - G(t) = O(e^{-ct}), \qquad c > 0,$$

we demonstrate the existence of a density function for $K(u)$, $u > 0$.

The investigation of the properties of the solutions of the non-linear integral equations requires a large number of ad hoc methods, patched together in no obvious fashion. Considering that we are dealing with non-linear processes, for which the treatment is as yet little standardized, there seems to be no alternative to this potpourri of methods.

The models treated in this paper are susceptible of generalization in several important directions. One may consider the more general case where the probability of transformation is dependent on the time of birth and on the number of contemporary particles. Then there are the problems of the distribution of ages, the number of transformations in a given interval, and so on. Finally, there is the case where there are particles of different types which give birth not only to those of the same type, but also to those of other types. The case of biological mutation is an example of this.

Finally, we may mention that the results of this paper are generalizations of those contained in a paper by one of the authors, Harris [3], where further references are given, and that several of the methods of the present paper are contained <u>in ovo</u> in this.

We should like to express our appreciation for the many helpful suggestions of the referee.

§2. Derivation of the Integral Equation.

The function $p_r(t)$, the probability that r particles exist at time t, satisfies

$$(1) \qquad p_r(t) = \int_0^t P[Z(t) = r|y]dG(y) + \delta_{1r}[1 - G(t)]$$

where δ_{1r} is the Kronecker delta function and $P[Z(t) = r|y]$ is the conditional probability that $Z(t) = r$ given that the initial particle was transformed at time y. It is clear that this conditional probability is given by

$$(2) \qquad P[Z(t) = r|y] = \sum_{i=0}^{r} p_i(t-y)p_{r-i}(t-y) .$$

Substituting (2) in (1), multiplying both sides by s^r, and summing from $r = 0$ to ∞ gives the following integral equation for

$F(s, t) = \sum_{r=0}^{\infty} p_r(t)s^r$:

$$(3) \qquad F(s, t) = \int_0^t F^2(s, t-y)dG(y) + s[1 - G(t)].$$

If $G(y) = 1 - e^{-ay}$, (3) may be reduced by differentiation with respect to t to an ordinary differential equation.

§3. Formal Definition of the Process.

In order to define formally the stochastic process with which we are dealing, we consider the space Ω of functions $Z(t)$, $0 \leq t < \infty$, whose values are nonnegative integers. First we must define the probabilities

$$P[Z(t_1) = r_1, Z(t_2) = r_2, \cdots, Z(t_k) = r_k] =$$

$$P_{r_1 r_2 \cdots r_k}(t_1, t_2, \cdots, t_k)$$

for every k nonnegative integers $r_1, \cdots, r_k$ and every k nonnegative numbers $t_1, \cdots, t_k$; $k = 1, 2, \cdots$. Once these definitions have been made, provided certain consistency relations hold, it follows from a theorem of Kolmogoroff [6] that a probability measure is uniquely defined on the Borel sets of Ω. By "Borel sets of Ω" we mean the Borel extension of the field of cylinder sets. A cylinder set is a set consisting of all functions $Z(t)$ such that $Z(t_i) \in S_i$, $i = 1, 2, \cdots, k$, where S_i is any set of nonnegative integers.

We define the probabilities $P_{r_1 \cdots r_k}(t_1, \cdots, t_k)$ by means of the generating functions

$$F_k(s_1, \cdots, s_k;\ t_1, \cdots, t_k) =$$

$$\sum P_{r_1 \cdots r_k}(t_1, \cdots, t_k)\, s_1^{r_1} \cdots s_k^{r_k}\,.$$

We define the F_k inductively, for $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k$, and agree to define

$$F_k\left[s_{\pi(1)}, \cdots, s_{\pi(k)};\ t_{\pi(1)}, \cdots, t_{\pi(k)}\right] =$$

$$F_k\left[s_1, \cdots, s_k;\ t_1, \cdots, t_k\right]$$

for any permutation $\pi(1), \cdots, \pi(k)$.

Having defined $F_1(s_1, t_1) = F(s_1, t_1)$ by means of (4), Section 1, and assuming that $F_1, F_2, \cdots, F_k$ have been defined, we define F_{k+1} by means of the equation

(1) $$F_{k+1}(s_1, \cdots, s_{k+1};\ t_1, \cdots, t_{k+1}) =$$

$$\int_0^{t_1} h\left[F_{k+1}(s_1, \cdots, s_{k+1};\ t_1 - y, \cdots, t_{k+1} - y)\right] dG(y)$$

$$+ s_1 \int_{t_1}^{t_2} h\left[F_k(s_2, \cdots, s_{k+1};\ t_2 - y, \cdots, t_{k+1} - y\right] dG(y)$$

$$+ \cdots + s_1 s_2 \cdots s_k \int_{t_k}^{t_{k+1}} h\left[F_1(s_{k+1};\ t_{k+1} - y)\right] dG(y)$$

$$+ s_1 s_2 \cdots s_{k+1}\left[1 - G(t_{k+1})\right].$$

The probabilistic reasons for the definitions (1) are analogous to those given in paragraph 2.

If $F_1, \cdots, F_k$ are known to be probability generating functions, it can be shown, following methods to be used in Section 4, that (1) determines uniquely among bounded functions the probability generating function F_{k+1}, provided $0 \leq t_1 \leq \cdots \leq t_{k+1}$.

It remains only to show the consistency of the probabilities that have been defined. This can be done by repeated application of the following type of argument. Set $s_1 = 1$ in (1), thus obtaining an equation for $F_{k+1}(1, s_2, \cdots)$ which is now identical with the equation used to define F_k; because of uniqueness, this implies $F_{k+1}(1, s_2, \cdots) = F_k(s_2, \cdots; t_2, \cdots)$.

§4. Existence and Uniqueness.

We shall demonstrate the following result:

Theorem 1: *Under the assumptions*

(1) (a) $dG \geq 0, \quad G(0+) = 0, \quad G(\infty) = 1,$

(b) G *continuous from the right*,

there exists a solution of the integral equation,

(2) $$F(s, t) = s(1 - G(t)) + \int_0^t F^2(s, t-y)\,dG(y),$$

which has the following properties:

(3) (a) $F(s, t) = \sum_{r=1}^{\infty} p_r(t)s^r$, <u>for</u> $|s| \leq 1$, <u>and all</u> $t \geq 0$.

(b) $F(s, 0) = s, \quad F(1, t) = 1,$

(c) $p_1(t) = 1 - G(t)$

$$p_r(t) = \int_0^t \sum_{j=1}^{r} p_j(t-y)p_{r-j}(t-y)dG(y), \qquad r \geq 2,$$

<u>whence, in particular,</u>

(d) $p_r(t) \geq 0$.

<u>The functions</u> $p_r(t)$, $r = 1, 2, \cdots$, <u>and</u> $F(s, t)$, $|s| \leq 1$, <u>are of bounded variation over every finite</u> t-<u>interval</u>.

<u>Furthermore, the above solution is the sole solution of</u> (2) <u>which is uniformly bounded for all</u> $t \geq 0$, <u>for each</u> s <u>in</u> $|s| \leq 1$.

<u>Proof</u>: Let us agree to the convention that a Stieltjes integral of the form $\int_a^b f(y)dg(y)$ is to be interpreted as $\int_{a+}^{b+} f(y)dg(y)$.

We shall utilize the method of successive approximations. Define

(4) $$F_0(s, t) = s(1 - G(t)),$$

$$F_{n+1}(s, t) = s(1 - G(t)) + \int_0^t F_n^2(s, t-y)dG(y), \qquad n \geq 0.$$

It follows readily by induction that

(5) $$|F_n| \leq 1, \quad t \geq 0, \quad |s| \leq 1, \quad n \geq 0.$$

For if the inequality, clearly true for $n = 0$, be assumed to hold for some n, we obtain from (4),

$$(6) \qquad |F_{n+1}| \leq \int_0^t dG(y) + 1 - G(t) = 1.$$

We first prove that the sequence $F_n(s, t)$ converges for all s in the interval $[0, 1]$ as follows. Each F_n is non-negative, and since $F_1 \geq F_0$, it follows by induction that $F_{n+1} \geq F_n$. Consequently, since for each s in $[0, 1]$, the sequence $\{F_n\}$ is monotone increasing in n and bounded, it converges for all s in $[0, 1]$. Call the limit function $F(s, t)$. Using the Lebesgue bounded convergence theorem, we see that

$$(7) \qquad F(s, t) = \int_0^t F^2(s, t-y)dG(y) + s(1-G(t)), \quad 0 \leq s \leq 1.$$

The sequence $\{F_n\}$ is thus a uniformly bounded sequence of analytic functions of s in the unit circle, $|s| \leq 1$, which converges on the segment $0 \leq s \leq 1$. It follows from Vitali's theorem, that the sequence converges uniformly to an analytic function in any closed region within the unit circle. It is not difficult to give a further argument showing that the convergence is uniform in and on the unit circle due to the positivity of the coefficients of the power series developments for $F_n(s, t)$. However, it is not entirely easy to generalize this line of proof to cover systems of equations of type (2), §3. For this reason, we present the following proof, which, although more pedestrian, is quickly applicable to the more general situation.

We have for $|s| \leq 1$, $n \geq 1$,

(8) $F_{n+1} - F_n =$

$$\int_0^t [F_n(s,\ t-y) - F_{n-1}(s,\ t-y)][F_n(s,\ t-y) + F_{n-1}(s,\ t-y)]dG(y),$$

whence

(9) $$|F_{n+1} - F_n| \leq 2\int_0^t |F_n(s,\ t-y) - F_{n-1}(s,\ t-y)|\ dG(y).$$

Restrict t, temporarily, to the interval $[0,\ T]$, where T is chosen so that

(10) $$2\int_0^T dG \leq b < 1.$$

This is possible since $G(0) = 0$. With this restriction (9) yields

(11) $$\sup_{0 \leq t \leq T} |F_{n+1} - F_n| \leq b \sup_{0 \leq t \leq T} |F_n - F_{n-1}|, \qquad n \geq 1.$$

From this, we obtain

(12) $$\sup_{0 \leq t \leq T} |F_{n+1} - F_n| \leq b^n, \qquad n \geq 1,$$

since $|F_1 - F_0| \leq 1$. Consequently, for $0 \leq t \leq T$, $|s| \leq 1$, the series

(13) $$\sum_{n=0}^{\infty} (F_{n+1} - F_n),$$

converges uniformly in t and s.

Let us now establish convergence in the interval $T \leq t \leq 2T$. We have, in this interval,

$$(14) \qquad F_{n+1} - F_n = \int_0^{t-T} + \int_{t-T}^{t} \left[F_n(s, t-y) - F_n(s, t-y)\right]\left[\cdots\right] dG(y)$$

and thus

$$(15) \qquad |F_{n+1} - F_n| \leq 2\int_0^{t-T} |F_n(s, t-y) - F_{n-1}(s, t-y)|\, dG(y)$$

$$+ 2\int_{t-T}^{t} |F_n(s, t-y) - F_{n-1}(s, t-y)|\, dG(y), \qquad n \geq 1.$$

We have already shown that $|F_{n+1} - F_n| \leq b^n$ in the interval $0 \leq t \leq T$. Hence, since $0 \leq t-y \leq T$ for $t-T \leq y \leq t$, we have

$$(16) \qquad |F_{n+1} - F_n| \leq 2\int_0^{t-T} |F_n(s, t-y) - F_{n-1}(s, t-y)|\, dG(y) + 2b^n, \qquad n \geq 1$$

$$|F_1 - F_0| \leq 1.$$

From this it follows easily that

$$(17) \qquad \sup_{T \leq t \leq 2T} |F_{n+1} - F_n| \leq (2n+1)b^n,$$

and, therefore, that the series $\sum_{n=0}^{\infty} (F_{n+1} - F_n)$ converges uniformly in s and t.

This same argument may be then repeated over the interval $[2T, 3T]$, and so on. From this we conclude that F_n converges uniformly to $F(s, t)$ over any fixed t-interval, for $|s| \leq 1$. An argument of similar type shows that (2) has only one bounded solution.

Referring to (3), we see that (a) follows from the fact that $F(s, t)$ is a limit of bounded sequences of power series with non-negative coefficients; the second part of (b) follows from the uniqueness of a bounded solution - in this case 1; and (c) follows by equating coefficients in (2).

It remains to show bounded variation. We have, for $t_1 > t_2 > \cdots > t_{n-1} > t_n$,

$$(18) \qquad F(s, t_k) - F(s, t_{k+1}) = \int_{t_{k+1}}^{t_k} F^2(s, t_k - y)\,dG(y) + \int_0^{t_{k+1}} \left[F^2(s, t_k - y) - F^2(s, t_{k+1} - y)\right] dG(y) + s\big(G(t_{k+1}) - G(t_k)\big).$$

Thus,

$$(19) \qquad |F(s, t_k) - F(s, t_{k+1})| \le 2\int_0^{t_{k+1}} |F(s, t_k - y) - F(s, t_{k+1} - y)|\,dG(y) + 2\big(G(t_k) - G(t_{k+1})\big).$$

Write

$$(20) \qquad \int_0^{t_k} = \int_0^{t_n} + \int_{t_n}^{t_{n-1}} + \cdots + \int_{t_{k+1}}^{t_k},$$

and

$$(21) \qquad V_N = \sup_{\{t_i\}} \sum_{k=1}^{N-1} |F(s, t_k) - F(s, t_{k+1})|, \qquad |s| \le 1.$$

Clearly $V_N \leq V_{N+1}$. Adding the inequalities of (19), using the decomposition of (20) and the notation of (21), we obtain

$$(22) \qquad \sum_{k=1}^{n-1} |F(s, t_k) - F(s, t_{k+1})| \leq 2V_n \int_0^{t_n} dG(y) + 2V_{n-1} \int_{t_n}^{t_{n-1}} dG(y)$$

$$+ \cdots + 2V_2 \int_{t_2}^{t_1} dG(y)$$

$$+ 2\big(G(t_1) - G(t_n)\big)$$

$$\leq 2V_n \int_0^{t_1} dG(y) + 2\big(G(t_1) - G(t_n)\big).$$

Therefore

$$(23) \qquad V_n \leq \left(2\int_0^{t_1} dG(y)\right) V_n + 2.$$

If we at first restrict ourselves to the interval $[0, T]$, where $2\int_0^T dG(y) = b < 1$, we deduce that $V_n \leq 2/(1-b)$, for all n, whence bounded variation. To establish bounded variation over $[T, 2T]$, we proceed as in (15) and (16).

Let us note that to extend our result to cover the more general equation

$$(24) \qquad F(s, t) = \int_0^t h\big(F(s, t-y)\big)dG(y) + s\big(1 - G(t)\big),$$

where

$$(25) \qquad h(x) = \sum_{n=2}^{\infty} a_n x^n, \qquad \sum_{n=2}^{\infty} a_n = 1, \qquad a_n \geq 0,$$

we need only add the additional condition $h'(1) < \infty$. The method of proof is then the same.

§ 5. Properties of the Moments.

Let us now discuss some further properties of the solutions. The first result is

Theorem 2. *Assuming*

(1) $$dG \geq 0, \qquad G(0+) = 0, \qquad G(\infty) = 1,$$

then all the moments

(2) $$m_k(t) = \sum_{n=1}^{\infty} n^k p_k(t), \qquad k = 1, 2, \cdots,$$

exist, and for fixed k, *and any* $\varepsilon > 0$,

(3) $$m_k(t) = O\left[e^{(ka+\varepsilon)t}\right],$$

as $t \longrightarrow \infty$, *where* a *is the positive root of*

(4) $$1 = 2\int_0^{\infty} e^{-at} dG(t).$$

For each k, $m_k(t)$ *is a nondecreasing function of* t.

Once the moments have been shown to exist, their nondecreasing character follows if we can show

(5) $$\mathrm{Prob}\left[z(t') \geq z(t)\right] = 1$$

for all $t' > t$. In order to prove (5), which is of course intuitively obvious, we consider the defining equations (1), Section 3, with $k = 1$. We need merely to show that if $F_2(s_1, s_2; t_1, t_2)$ is written as a power series in s_1, the coefficient of s_1^k, for each integer k, contains s_2^k as a factor. This is readily done by successive derivation of (1), Section 3, with respect to s_1 at $s_1 = 0$, using induction on k.

Now consider $m_1(t)$. For $|s| < 1$, we have

$$(6) \qquad F'(s, t) = 2\int_0^t F'(s, t-y)F(s, t-y)dG(y) + 1-G(t).$$

(' denotes differentiation with respect to s.) Hence

$$(7) \qquad |F'(s, t)| \leq 2\int_0^t |F'(s, t-y)|dG(y) + 1 .$$

Set $|F'(s, t)| = e^{bt}v(s, t)$, where $b > a$, a being defined by (4). Then,

$$(8) \qquad v(s, t) \leq 2\int_0^t v(s, t-y)e^{-by}dG + e^{-bt},$$

and consequently,

$$(9) \qquad \sup_{0\leq t\leq T} v(s, t) \leq \sup_{0\leq t\leq T} v(s, t)\left(2\int_0^\infty e^{-by}dG\right) + 1,$$

or

$$(10) \qquad \sup_{0\leq t\leq T} v(s, t) \leq 1/ \left(1 - 2\int_0^\infty e^{-by}dG\right).$$

The bound is independent of s and this together with the fact that

$p_r(t) \geq 0$, implies (3) for the case $k = 1$, since $m_1(t) = F'(1, t)$.

Since $F^{(2)}(1, t) = E[Z(t)]^2 - E[Z(t)]$, and in general

(11) $$F^{(k)}(1, t) = E[Z(t)]^k + \text{expected value of powers less than } k \text{ of } Z(t),$$

it suffices to consider $F^{(k)}(1, t)$ rather than $m_k(t)$.

The second derivative for $|s| < 1$ satisfies

(12) $$F^{(2)}(s, t) = 2\int_0^t F(s, t-y)F^{(2)}(s, t-y)dG(y) + 2\int_0^t [F'(s, t-y)]^2 dG(y).$$

Using the result for $k = 1$ and the method above, we easily show that

$$\sup_{0 \leq t < \infty} |F^{(2)}(s, t)|$$

has a bound independent of s for $|s| < 1$. Now

$$F^{(2)}(s, t) = \sum_{r=1}^{\infty} (r^2 - r)p_r(t)s^{r-2}$$

and since $r^2 - r$ is positive for $r \geq 2$, the theorem follows for $k = 2$. The process may now be continued and the general result obtained by induction.

Before proceeding to the question of the asymptotic behavior of the moments we prove the following

<u>Lemma</u> 1. <u>If</u> $v(t)$ <u>satisfies the equation</u>

(12) $$v(t) = \int_0^t v(t-y)dH(y) + K(t)$$

and is bounded on every finite interval, where

(13) (a) $dH \geq 0, \quad H(0+) = 0, \quad H(\infty) = \alpha < 1$

(b) $|K(t)| \leq c_1, \quad 0 \leq t < \infty$

(c) $K(t) \longrightarrow c_2$ *as* $t \longrightarrow \infty,$

then

(14) $v(t) \longrightarrow c_2/(1-\alpha), \qquad t \longrightarrow \infty.$

Furthermore, if $|K(t) - c_2| = O(e^{-\varepsilon t})$ *and* $|\alpha - H(t)| = O(e^{-\varepsilon t})$, $\varepsilon > 0$, *as* $t \longrightarrow \infty$, *then there is a* $\delta > 0$ *such that*

(15) $|v(t) - c_2/(1-\alpha)| = O(e^{-\delta t}), \qquad t \longrightarrow \infty.$

Proof. First assume $c_2 = 0$. Since $v(t)$ is bounded for $0 \leq t \leq T$, we have from (12)

$$\sup_{0 \leq t \leq T} |v(t)| \leq H(T) \sup_{0 \leq t \leq T} |v(t)| + c_1,$$

or

$$\sup_{0 \leq t \leq T} |v(t)| \leq \frac{c_1}{1 - H(T)} \leq \frac{c_1}{1-\alpha}.$$

Thus $|v(t)| \leq c_1/(1-\alpha)$, $0 \leq t < \infty$. Using (12) again, let

$$\bar{v}(t) = \sup_{\tau \geq t} |v(\tau)|,$$

$$\bar{K}(t) = \sup_{\tau \geq t} |K(\tau)|;$$

then for any $T > 0$ and integer n,

$$\text{(16)} \qquad \bar{v}[(n+1)T] \leq \int_0^T \bar{v}[(n+1)T - y]dH(y)$$

$$+ \int_T^{(n+1)T} \bar{v}[(n+1)T - y]dH(y) + \bar{K}[(n+1)T]$$

$$\leq \alpha\bar{v}(nT) + \frac{[\alpha - H(T)]c_1}{1-\alpha} + \bar{K}[(n+1)T] .$$

Let k be a positive integer. Since $|\bar{v}(t)| \leq \frac{c_1}{1-\alpha}$ and is monotone decreasing, the inequality

$$\text{(17)} \qquad \bar{v}\big((n+1)T\big) \geq \bar{v}\big(nT\big) - \frac{c_1}{\kappa(1-\alpha)}$$

must hold for at least one value of n between 0 and $k-1$, say $n = n_0$. Combining (17), for $n = n_0$, with (16) gives

$$\text{(18)} \qquad \bar{v}(n_0T) \leq \frac{1}{1-\alpha}\left\{\frac{c_1}{\kappa(1-\alpha)} + \frac{c_1[\alpha - H(T)]}{1-\alpha} + \bar{K}[(n_0+1)T]\right\} .$$

Since $\bar{v}(n_0T) \geq \bar{v}(\kappa T)$, $\bar{K}[(n_0+1)T] \leq \bar{K}(T)$, we have

$$\text{(19)} \qquad \bar{v}(\kappa T) \leq \frac{1}{1-\alpha}\left\{\frac{c_1}{\kappa(1-\alpha)} + \frac{c_1[\alpha - H(T)]}{1-\alpha} + \bar{K}(T)\right\}, \qquad \kappa = 1, 2, \cdots.$$

From (19) follows (14) for the case $c_2 = 0$. The general case is proved by making the change of variable

$$v^*(t) = v(t) - c_2/(1-\alpha) .$$

The proof used here, although lengthy, shows, by means of (19), that for each $\underline{t}$, $|v(t) - c_2/(1-\alpha)|$ can be bounded by a quantity which depends only

on $|\alpha - H(t)|$ and $\sup_{t \le T} |K(T) - c_2|$, a fact we shall need later.

If $|K(t) - c_2|$ and $|\alpha - H(t)|$ are both $O(e^{-\varepsilon t})$, choose λ, $0 < \lambda < \varepsilon$, small enough that $\int_0^\infty e^{\lambda t} dH(t) < 1$. We can then apply (14) to the function

$$v^*(t) = e^{\lambda t}[v(t) - c_2/(1 - \alpha)]$$

obtaining (15). Again, the bound $O(e^{-\delta t})$ in (15) depends only on $|\alpha - H(t)|$ and $\sup_{t \le T} |c_2 - K(T)|$.

We are now ready to consider the asymptotic behavior of the moments. Setting $s = 1$ in (6) we obtain

$$(20) \qquad m_1(t) = 2\int_0^t m_1(t-y)dG(y) + 1 - G(t).$$

This is the familiar integral equation of renewal theory, for which general results, including theorems on the asymptotic behavior of $m_1(t)$, have been obtained by Feller [2] and Täcklind [7]. However, there are special conditions satisfied in the present problem which make more precise results possible. On the other hand, as Feller has shown, still more specialized assumptions on G(t) would enable us to use the method of Lotka to expand $m_1(t)$ as a series of exponentials.

It is well known that the solution of (20) may be written as

$$m_1(t) = 1 + \sum_{n=1}^{\infty} 2^{n-1} G_n(t),$$

the series converging uniformly in every finite t-interval, where

$$G_1(t) = G(t), \quad G_{n+1}(t) = \int_0^t G_n(t-y)dG(y), \quad n = 1, 2, \cdots .$$

Thus continuity of $G(t)$ implies continuity of $m_1(t)$, and similar expressions for the higher moments show them to be continuous if $G(t)$ is. Furthermore, if $G(t)$ is a step function the moments are step functions with discontinuities only at points of the form $t_1 + t_2 + \cdots + t_n$, where $t_1, t_2, \cdots, t_n$ are points of discontinuity of $G(t)$.

For convenience we call $G(t)$ a <u>simple step function</u> if it is a step function, all of whose steps occur at integral multiples of some unit.

<u>Theorem 3. Assume</u> $dG \geq 0$, $G(0+) = 0$, $G(\infty) = 1$, <u>and that</u> $G(t)$ <u>is not a simple step function, then the solution of</u> (20) <u>satisfies</u>

(21) $$m_1(t) \sim n_1 e^{at}, \qquad t \longrightarrow \infty,$$

$$n_1 = 1 \Big/ \left[4a \int_0^\infty t\, e^{-at} dG(t)\right]$$

<u>where</u> a <u>is defined by</u> (4).

By virtue of (3), Theorem 2, we can take Laplace transforms of both sides of (20), obtaining, for $Re(s) > a$,

(22) $$\int_0^\infty m_1(t)e^{-st}dt = \int_0^\infty [1 - G(t)]e^{-st}dt \Big/ \left[1 - 2\int_0^\infty e^{-st}dG(t)\right]$$

$$= \frac{1 - \Psi(s)}{s[1 - 2\Psi(s)]}$$

where

$$\Psi(s) = \int_0^\infty e^{-st} dG(t), \qquad Re(s) \geq 0.$$

The right side of (22), which can be extended analytically to the region $Re(s) > 0$, has a simple pole at $s = a$, and no other singularities on the line $Re(s) = a$, since $G(t)$ is not a simple step function. Moreover, we have shown that $m_1(t)$ is nondecreasing. We can therefore apply a Tauberian theorem of Ikehara [8], which immediately gives Theorem 3.

<u>Theorem</u> 4a. <u>Assume</u> $G(t) = \int_0^t g(y)dy$, $g(y) \geq 0$, $G(\infty) = 1$, <u>and that</u> $\Psi(s) = \int_0^\infty e^{-st} g(t)dt$ <u>satisfies the condition that for every</u> b, $0 < b < a$,

$$(23) \qquad \int_{-\infty}^{\infty} \frac{|\Psi(b + iy)|}{1 + |y|} dy < \infty .$$

<u>Then</u> $m_1(t)$ <u>satisfies</u>

$$(24) \qquad m_1(t) = n_1 e^{at}\left[1 + O(e^{-\varepsilon t})\right], \qquad \varepsilon > 0, \quad t \longrightarrow \infty,$$

<u>where</u> n_1 <u>is defined in</u> (21).

<u>Theorem</u> 4b. <u>If</u> $dG \geq 0$, $G(0+) = 0$, $G(\infty) = 1$, <u>and</u> $G(t)$ <u>is a simple step function, the smallest step being at</u> Δ, <u>then for each</u> α, $0 < \alpha < \Delta$, and integer n,

$$m_1(\alpha + n\Delta) = \frac{e^{an\Delta}\left[1 + O(e^{-n\varepsilon})\right]}{4(1 - e^{-a\Delta})\sum_1^\infty k\, b_k e^{-ka\Delta}}, \qquad \varepsilon > 0, \quad n \longrightarrow \infty,$$

where b_k *is the magnitude of the jump of* $G(t)$ *at* $t = k\Delta$ *and* $b_1 > 0$.

Theorem 4c. *If* $dG \geq 0$, $G(0+) = 0$, $G(\infty) = 1$, $G(t) = \int_0^t g_1(y)dy + G_2(t)$, $g_1(y) \geq 0$, $\int_0^\infty g_1(y)dy > 0$, $\varphi_1(s) = \int_0^\infty e^{-st} g_1(t)dt$ *satisfies* (23), *and* $G_2(t)$ *is a step function*: *then* (24) *holds*.

Before giving the proof we remark that there are two simple conditions, either of which insures that (23) holds. If for some d, $0 < d < a$, the function $g(t)e^{-dt}$ is of bounded total variation on $(0, \infty)$, then (23) follows from integration by parts. Another condition, which permits $g(t) = O(t^{-\varepsilon})$, $t \longrightarrow 0$, for any $0 < \varepsilon < 1$, is the assumption that for some $p > 1$,

$$(25) \qquad \int_0^\infty g(t)^p dt < \infty.$$

For, applying Hölder's inequality, with $0 < b < a$,

$$(26) \qquad \int_{-\infty}^\infty \frac{|\varphi(b + iy)|}{1 + |y|} dy \leq \left(\int_{-\infty}^\infty |\varphi(b + iy)|^{p'} dy\right)^{\frac{1}{p'}} \left(\int_{-\infty}^\infty \frac{dy}{(1 + |y|^p)}\right)^{\frac{1}{p}},$$

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Since $\varphi(b + iy)$ is the Fourier transform of $e^{-bt}g(t)$, we have, using the Hausdorff-Young inequality for Fourier integrals,

$$(27) \qquad \left(\int_{-\infty}^\infty |\varphi(b + iy)|^{p'} dy\right)^{\frac{1}{p'}} \leq c(p)\left(\int_{-\infty}^\infty e^{-pbt} g(t)^p dt\right)^{\frac{1}{p}}.$$

From (25), (26), and (27) we get (23).

The proof of Theorem 4a is by the method of residues, used with what Doetsch calls the "indirect Abelian" method. Since $m_1(t)$ is continuous (and nondecreasing) we have from (22) for any $t > 0$, $b' > a$,

$$(28) \qquad m_1(t) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{b'-iT}^{b'+iT} e^{st} \left\{ \frac{1 - \varphi(s)}{s[1 - 2\varphi(s)]} \right\} ds .$$

The function $1 - 2\varphi(s)$ has a simple zero at $s = a$ and vanishes nowhere else on the line $Re(s) = a$. Moreover, since $\varphi(s)$ is the Laplace transform of an absolutely continuous distribution we have

$$\lim_{y\to\pm\infty} |\varphi(b + iy)| = 0$$

uniformly in b for $0 \le b \le a$. We can therefore find b, $0 < b < a$, such that $1 - 2\varphi(s)$ has no zeros, except at $s = a$, in the strip $b \le Re(s) \le a$ (clearly there are no zeros for $Re(s) > \underline{a}$) and is uniformly bounded away from 0 on the line $Re(s) = b$. Then, using Cauchy's theorem for the rectangle $b \pm iT$, $b' \pm iT$, we obtain (the integrals along the horizontal sides of the rectangle clearly $\to 0$ as $T \to \infty$)

$$(29) \qquad m_1(t) = n_1 e^{at} + \lim_{T\to\infty} \frac{1}{2\pi i} \int_{b-iT}^{b+iT} e^{st} \left\{ \frac{1}{s} + \frac{\varphi(s)}{s(1 - 2\varphi(s))} \right\} ds$$

$$= n_1 e^{at} + e^{bt} + \lim_{T\to\infty} \frac{1}{2\pi i} \int_{b-iT}^{b+iT} e^{st} \frac{\varphi(s)ds}{s(1 - 2\varphi(s))}$$

where n_1 is defined by (21) and $n_1 e^{at}$ is the residue of the integrand at $s = a$. The conditions imposed on $\varphi(s)$ in Theorem 4a insure that

the integral on the right side of (29) is $O(e^{bt})$, and this concludes the proof of 4a.

To prove Theorem 4b we note that in this case the function $\varphi(s)$ has the period $2\pi i/\Delta$,

$$\varphi(s) = \int_0^\infty e^{-st}dG(t) = \sum_{k=1}^\infty b_k e^{-k\Delta s},$$

$$b_k \geq 0, \qquad b_1 > 0, \qquad \sum_1^\infty b_k = 1.$$

Thus $1 - 2\varphi(s)$ has simple zeros at the points $a \pm 2\pi ir/\Delta$, $r = 0, 1, \cdots$, and because of periodicity we can find b, $0 < b < a$, such that $1 - 2\varphi(s)$ is uniformly bounded away from 0 on the line $\mathrm{Re}(s) = b$, and such that if $b' > a$, the only zeros in the strip $b \leq \mathrm{Re}(s) \leq b'$ are those on $\mathrm{Re}(s) = b$. Moreover, $\varphi(s)$ and $[1 - \varphi(s)] / [1 - 2\varphi(s)]$ are twice continuously differentiable on $\mathrm{Re}(s) = b$. Thus

$$\frac{1 - \varphi(b + iy)}{1 - 2\varphi(b + iy)} = \sum_{k=-\infty}^\infty c_k e^{ik\Delta y}, \qquad c_k = O(1/k^2), \quad k \to \pm\infty.$$

Using Cauchy's method as before, we obtain the principal term in Theorem 4b by summing the residues on $\mathrm{Re}(s) = a$, with the remainder

$$(30) \qquad \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{bt+iyt}}{b + iy} \left(\sum_{k=-\infty}^\infty c_k e^{ik\Delta y}\right) dy.$$

Since $\sum_{k=-\infty}^\infty |c_k| < \infty$ and since

(31) $$\lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{ity} e^{ik\Delta y}}{b + iy}\, dy = \begin{cases} e^{-b(t+k\Delta)}, & t + k\Delta > 0 \\ 0, & t + k\Delta < 0 \end{cases},$$

and since furthermore the convergence in (31) is bounded uniformly in t and k provided $t + k\Delta$ is bounded away from 0, we can interchange summation and integration in (30), obtaining

(32) $$e^{bt} \sum_{k=-[t/\Delta]}^{\infty} c_k e^{-b(t+k\Delta)} .$$

This proves Theorem 4b.

If the conditions of Theorem 4c hold, the Laplace transform of $m_1(t)$ can be written

(33) $$\frac{1}{s} + \frac{\varphi_1(s)}{s[1 - 2\varphi(s)]} + \frac{\varphi_2(s)}{s[1 - 2\varphi(s)]} ,$$

where

$$\varphi_1(s) = \int_0^\infty e^{-st} g_1(t)dt,$$

$$\varphi_2(s) = \int_0^\infty e^{-st} dG_2(t) = \sum_{k=1}^{\infty} d_k e^{-\lambda_k t} ,$$

$$\lambda_k > 0, \quad d_k > 0, \quad \sum d_k < 1.$$

As in Theorem 4a, $1 - 2\varphi(s)$ has a simple zero at $s = a$ and no other zeros on $\mathrm{Re}(s) = a$, and $\varphi_1(b + iy) \longrightarrow 0$ as $y \longrightarrow \pm\infty$, uniformly in b for $0 \leq b \leq a$. We can thus find b, $0 < b < a$, so that $s = a$ is the only zero of $1 - 2\varphi(s)$ in the strip $b \leq \mathrm{Re}(s) \leq a$. By our

assumptions, $\Psi_1(s)/[s(1-2\Psi(s))]$ is absolutely integrable on $Re(s) = b$. As regards the third term in (33), we can write

$$(34) \qquad \frac{\Psi_2}{s(1-2\Psi)} = \frac{\Psi_2}{s(1-2\Psi_2)} + \frac{2\Psi_1\Psi_2}{s(1-2\Psi_2)(1-2\Psi)} .$$

The second term on the right side of (34) is absolutely integrable on $Re(s) = b$. If $a - b$ is sufficiently small and positive, we have

$$|2\Psi_2(b+iy)| < 1$$

uniformly for $-\infty < y < \infty$. Thus, since Ψ_2 is an absolutely convergent series we have, for $a - b$ sufficiently small and positive,

$$\frac{\Psi_2(s)}{1-2\Psi_2(s)} = \sum_{k=0}^{\infty} f_k e^{-\mu_k s}, \qquad \sum_{k=0}^{\infty} |f_k| < \infty, \quad \mu_k \geq 0.$$

The proof of Theorem 4c is then carried out using the procedure of 4a and 4b.

It is now easy to get the asymptotic form of the higher moments, using Lemma 1. Let $u_k(t) = F^{(k)}(1, t)$. Then, as remarked earlier, we can consider the asymptotic behavior of $u_k(t)$ rather than $m_k(t)$. Setting $s = 1$ in (12) gives

$$(35) \qquad u_2(t) = 2\int_0^t u_2(t-y)dG(y) + 2\int_0^t [m_1(t-y)]^2 dG(y).$$

Put $u_2(t) = e^{2at}v_2(t)$. Then

$$(36) \qquad v_2(t) = 2\int_0^t v_2(t-y)e^{-2ay}dG(y)$$

$$+ 2\int_0^t [e^{-a(t-y)}m_1(t-y)]^2 e^{-2ay}dG(y) .$$

Now $2\int_0^\infty e^{-2ay}dG(y) < 2\int_0^\infty e^{-ay}dG(y) = 1$, and if $G(y)$ is not a simple step function we have, from Theorem 4a, $e^{-at}m_1(t) \longrightarrow n_1$ as $t \longrightarrow \infty$, whence

$$\lim_{t\to\infty} 2\int_0^t [e^{-a(t-y)}m_1(t-y)]^2 e^{-2ay}dG(y) = 2n_1^2\int_0^\infty e^{-2ay}dG(y).$$

Therefore, since from Theorem 2 $v_2(t)$ is bounded on every finite interval, we can apply Lemma 1 to (36) obtaining

<u>Theorem</u> 5. <u>Under the conditions of Theorem</u> 3,

$$\lim_{t\to\infty} m_2(t)e^{-2at} = \frac{2n_1^2\int_0^\infty e^{-2ay}dG(y)}{1-2\int_0^\infty e^{-2ay}dG(y)},$$

<u>where</u> n_1 <u>is defined by</u> (21), <u>and</u> a <u>is defined by</u> (4).

Results corresponding to Theorems 4a, b, and c can be obtained in the same way. This will be seen more generally in the next section when we consider the mixed moment $E[Z(t)Z(t+h)]$.

§6. <u>Mean Square Convergence of</u> $Z(t)/m_1(t)$.

<u>Theorem</u> 6. <u>Under the hypotheses of Theorem</u> 3 <u>the random variable</u> $Z(t)/m_1(t)$ <u>converges in mean square to a random variable</u> w <u>as</u> $t \longrightarrow \infty$. <u>Under the hypotheses of Theorem</u> 4b, $Z(n\Delta)/EZ(n\Delta)$ <u>converges with probability one to a random variable</u> w, <u>and also in mean square</u>.

Theorem 6 of course implies also convergence in probability to w.

Define $w(t)$ by

$$w(t) = Z(t) / \left[n_1 e^{at}\right] .$$

It is clearly sufficient to show that $w(t)$ converges in m.s., which we do by showing that

$$(1) \qquad \lim_{t,t' \to \infty} E\left[w(t') - w(t)\right]^2 = 0.$$

Differentiation of (1), Section 3, with $\kappa = 1$, gives

$$(2) \qquad \left.\frac{\partial^2 F_2(s_1, s_2; t_1, t_2)}{\partial s_1 \partial s_2}\right|_{s_1=s_2=1} = E\left[Z(t_1)Z(t_2)\right]$$

$$= m_2(t_1, t_2) = 2\int_0^{t_1} m_2(t_1-y, t_2-y)dG(y)$$

$$+ 2\int_0^{t_1} m_1(t_1-y)m_1(t_2-y)dG(y)$$

$$+ 2\int_{t_1}^{t_2} m_1(t_2-y)dG(y) + 1 - G(t_2).$$

Set $t_1 = t$, $t_2 = t+h$, $h \geq 0$, and let $m_2(t, t+h) = e^{ah}e^{2at}u(t, h)$. Then (2) becomes

$$(3) \qquad u(t, h) = 2\int_0^t u(t-y, h)e^{-2ay}dG(y) + c_3 + o(1) \text{ as } t \to \infty,$$

using the known asymptotic behavior of $m_1(t)$. A routine estimation shows that the o(1) in (3) is independent of <u>h</u>. The constant c_3 is given by

(4) $$c_3 = 2n_1^2 \int_0^\infty e^{-2ay} dG(y).$$

It follows now from Lemma 1, Section 5, that

(5) $$\lim_{t \to \infty} u(t, h) = \frac{c_3}{1 - 2\int_0^\infty e^{-2ay} dG(y)} = n_2$$

uniformly in h. Thus (see the remark following (19), Section 5)

(6) $$E[Z(t)Z(t + h)] = n_2 e^{ah} e^{2at}[1 + o(1)], \qquad h \geq 0.$$

Equation (1) now follows from (6) and from Theorem 5.

If the conditions of Theorem 4a or 4c hold, the o(1) in (3) goes to zero exponentially and it follows from (15), Section 5 (under Lemma 1), that there is an $\varepsilon > 0$ such that the o(1) in (6) is $O(e^{-\varepsilon t})$. From (6) and the remarks following Theorem 5, there exists a (different) $\varepsilon > 0$ such that (uniformly in $h > 0$)

(7) $$E[w(t + h) - w(t)]^2 = O(e^{-\varepsilon t}), \qquad t \to \infty,$$

which implies also that

(8) $$E[w - w(t)]^2 = O(e^{-\varepsilon t}), \qquad t \to \infty.$$

From (8) we have

<u>Theorem 6a</u>. <u>Under the hypotheses of Theorem</u> 4a <u>or</u> 4c <u>the random variables</u> w(nh), n = 1, 2, ⋯, <u>for any</u> h > 0, <u>converge with probability</u> 1 <u>to the random variable</u> w.

For

$$(9) \qquad \sum_{n=1}^{\infty} E[w - w(nh)]^2 < \infty,$$

which implies almost-everywhere convergence. Rather than the sequence nh we could consider any sequence t_n such that $\sum_n e^{-\varepsilon t_n} < \infty$.

If the conditions of Theorem 4b hold, similar arguments show that $Z(n\Delta) / EZ(n\Delta)$ converges with probability 1.

We define the moment-generating functions of $w(t)$ and w by

$$\phi(s, t) = E\left\{\exp[sZ(t)e^{-at}/ n_1]\right\}$$

$$= F\left\{\exp[se^{-at}/n_1], t\right\},$$

$$\phi(s) = E e^{sw}, \qquad \mathrm{Re}(s) \leq 0.$$

Since $w(t)$ converges in mean square to w the function $\phi(s, t)$ converges to $\phi(s)$ for each s whose real part is nonnegative. (Replace t by $n\Delta$ if $G(t)$ is a simple step function.) Moreover, $\phi(s)$ is continuous and the derivatives $\frac{d\phi(s, t)}{ds}$ are uniformly bounded for $\mathrm{Re}(s) \geq 0$, being bounded by $E\,w(t)$. Thus $\phi(s, t) \longrightarrow \phi(s)$ uniformly in every bounded portion of the half plane $\mathrm{Re}(s) \leq 0$. Now replace s by $\exp[se^{-at}/ n_1]$ in (7), Section 4; letting $t \longrightarrow \infty$ gives, because of uniformity of convergence,

$$(10) \qquad \phi(s) = \int_0^{\infty} \phi^2(se^{-ay})dG(y), \qquad \mathrm{Re}(s) \leq 0.$$

Summarizing we have

Theorem 7. *Assume* $G(0+) = 0$, $G(\infty) = 1$, $dG \geq 0$. *The moment generating function* $\phi(s)$ *of the random variable* w *satisfies* (10).

(If $G(t)$ is a simple step function $\underline{w}$ is l.i.m. $Z(n\Delta) / EZ(n\Delta)$; otherwise $\underline{w}$ = l.i.m. $Z(t) / EZ(t)$.)

§ 7. Analyticity of $\phi(s)$.

Theorem 8. *There is a unique function* $\phi(s)$ *which is analytic in some neighborhood of* $s = 0$, *and satisfies* (10) *of Section* 6, *with* $\phi(0) = \phi'(0) = 1$, *provided that* $G(0+) = 0$, $\int_0^\infty dG = 1$, $dG \geq 0$.

First, assume

$$(1) \qquad \phi(s) = \sum_{n=0}^{\infty} c_n s^n, \qquad c_0 = 1, \quad c_1 = 1.$$

Substitution in the integral equation gives, for $n \geq 2$,

$$(2) \qquad c_n = \sum_{k+j=n} c_k c_j \int_0^\infty e^{-nay} dG(y),$$

or, setting $I_n = \int_0^\infty e^{-nay} dG(y)$,

$$(3) \qquad c_n = \left(\sum_{\substack{k+j=n \\ k,j \geq 1}} c_k c_j \right) I_n / (1 - 2I_n).$$

Since $1 - 2I_2 > 0$, and $I_n \longrightarrow 0$ as $n \longrightarrow \infty$ (because $G(0+) = 0$), we can pick n_0 so that for $n \geq n_0$, $I_n / (1 - 2I_n) < 1$. We now show that there are constants d and A such that

(4) $$c_n \leq Ad^n/n^2, \qquad n \geq 1.$$

Assume that the inequality (4) holds for $1 \leq n \leq N$, where $N \geq n_0$. Clearly this can be accomplished with an A which satisfies

(5) $$32A \sum_{k=1}^{\infty} \frac{1}{k^2} < 1$$

by taking d sufficiently large. Then

(6) $$c_{N+1} \leq A^2 d^{N+1} \sum_{k=1}^{N} \frac{1}{k^2(N+1-k)^2}$$

$$\leq \frac{8A^2 d^{N+1}}{N^2} \sum_{k=1}^{\left[\frac{N}{2}+1\right]} \frac{1}{k^2} \leq \frac{32A^2 d^{N+1}}{(N+1)^2} \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Combining (5) and (6) gives (4). It can now be verified that the series defined by (1) satisfies the conditions of the theorem uniquely.

§8. Asymptotic Behavior of $\phi(it)$ as $t \longrightarrow \pm\infty$.

Since the substitution s = it converts the Laplace transform into a Fourier-Stieltjes transform, the cumulative distribution of w may be evaluated by means of the formula

(1) $$K(x+h) - K(x) = \lim_{T \to \infty} \int_{-T}^{T} \left(1 - e^{-it_1 h}\right) e^{-it_1 x} \phi(it_1)/(2\pi i t_1)\, dt_1 .$$

whenever x and x + h are continuity points of K.

There are several theorems available which relate the behavior of K(x) to the asymptotic behavior of $\phi(it)$. Hence it is of some theoretical interest to study the function $\phi(it)$ as $t \longrightarrow \pm\infty$.

Furthermore, it is practically important, when actually determining the cumulative distribution by **numerical** methods, to know how much of an error is committed by omitting part of the range of integration.

The result we wish to prove is

<u>Theorem</u> 9. <u>As</u> $t \longrightarrow \pm\infty$

(2) $$|\phi(it)| = O\left(|t|^{-c/a}\right),$$

<u>provided</u> $\int_x^{\infty} dG = O(e^{-cx})$, $c > 0$, <u>as</u> $x \longrightarrow \infty$, $G(0+) = 0$, $G(\infty) = 1$, $dG \geq 0$, <u>and</u> $G(t)$ <u>is not a step function with one step</u>.

The proof is rather long, and we shall break it up into a succession of lemmas.

<u>Lemma</u> 1. <u>As</u> $t \longrightarrow \pm\infty$, $\phi(it) \longrightarrow 0$. (<u>The assumption</u> $\int_x^{\infty} dG(y)$ <u>is not required for this</u>.)

<u>Proof</u>: Since

(3) $$\phi(it) = \int_{0-}^{\infty} e^{ixt} dK(x),$$

for small $|t|$, we have the expansion

(4) $$\phi(it) = 1 + it - \frac{t^2}{2}\int_{0-}^{\infty} x^2 dK(x) + o(|t|^2)$$

(since $\int_{0-}^{\infty} x\,dK(x) = 1$). Hence, by a familiar argument based upon the Schwarz inequality, for $|t|$ small, $|\phi(it)|^2 < 1$ if $t \neq 0$. (The strict inequality $(Ew)^2 < Ew^2$ is a consequence of our assumption that $G(t)$ is not a step function with one step.)

We remark that some such discussion as that above is necessary to distinguish the function $\phi(s)$ we are actually interested in from other functions satisfying the same equation, and in particular from the function 1, which is a solution not approaching zero as $t \longrightarrow \pm\infty$.

It is convenient to show first that

$$\limsup_{t \to \infty} |\phi(it)| < 1.$$

Suppose the contrary. Let $t_3 > 0$ be such that $|\phi(it_3)| < 1 - d$, where $d > 0$, and such that $|\phi(it)| < 1$ for $0 < t < t_3$, and let t_1 and t_2 be the first points to the left and right of t_3 for which $|\phi(it_1)| = |\phi(it_2)| = 1 - d$. Pick $A = \frac{1}{a} \log \frac{t_2}{t_1}$. Then

$$\phi(it_2) = \int_0^A \phi^2(it_2 e^{-ay}) dG(y) + \int_A^\infty \phi^2 dG(y)$$

and

$$1 - d = |\phi(it_2)| \leq (1 - d)^2 G(A) + 1 - G(A)$$

whence

$$(5) \qquad (2 - d)G(A) \leq 1.$$

Now let t_3 remain fixed while $d \to 0$; then $t_1 \to 0$ while t_2 increases, so that $A \to \infty$, $G(A) \to 1$, and (5) cannot continue to hold.

Thus we can suppose that $|\phi(it)| < 1 - d$ for $t \geq t_1$. Take B large enough so that $1 - G(B) < \varepsilon$, and t large enough to have $t e^{-aA} \geq t_1$, where A is, as above, $\log(t_2/t_1)/a$. Then

$$|\phi(it)| < (1 - d) \int_0^A |\phi(ite^{-ay})| dG(y) + \varepsilon$$

or, letting $\Psi(t) = \sup_{\tau \geq t} |\phi(i\tau)|$,

$$(6) \qquad \Psi(t) \leq (1 - d)G(A)\, \Psi(te^{-aA}) + \varepsilon .$$

From (6) it follows, in the manner of previous proofs, that $\Psi(t) \to 0$ as $t \to \infty$. A similar argument shows $\phi(it) \to 0$ as $t \to -\infty$. (We have purposely not made use of the analyticity of $\phi(s)$, since in the more general case where fission is not binary $\phi(s)$ may not be analytic.)

So far no condition has been imposed on the rate of approach of $G(t)$ to 1 as $t \longrightarrow \infty$. We now show that by imposing suitable conditions on this approach, we can derive explicit bounds for $|\phi(it)|$ as $t \longrightarrow \pm\infty$.

__Lemma__ 2. __If as__ $x \longrightarrow \infty$,

$$(7) \qquad \int_x^\infty dG = O(e^{-cx}), \qquad c > 0,$$

__then as__ $t \longrightarrow \pm\infty$,

$$(8) \qquad |\phi(it)| = O(|t|^{-d}),$$

__for some__ $d > 0$.

Choose $A = (\log t)/2a$. Then from the integral equation we derive

$$(9) \qquad |\phi(it)| \le \int_0^A |\phi^2(ite^{-ay})|dG(y) + 1 - G(A).$$

From the definition of A, we have

$$(10) \qquad \epsilon(t) = 1 - G(A) = 1 - G[(\log t)/2a] .$$

Once again set

$$(11) \qquad \Psi(t) = \sup_{\tau \ge t} |\phi(i\tau)| .$$

With this notation, from (9) we derive

(12) $$\varphi(t) \leq \varphi^2(\sqrt{t}) + b(t).$$

Under the assumption of (7), we have, for large t,

(13) $$\varphi(t) \leq \varphi^2(\sqrt{t}) + \exp(-c \log t/2a).$$

Hence

(14) $$\varphi\left(t^{2^{n+1}}\right) \leq \varphi^2\left(t^{2^n}\right) + \exp(-2^n c \log t/a)$$

or, setting $u_n = \varphi\left(t^{2^n}\right)$

(15) $$u_{n+1} \leq u_n^2 + \exp(-2^n c \log t/a).$$

Applying the inequality $(u+v)^2 \leq 2(u^2+v^2)$ after squaring both sides, the result is

(16) $$u_{n+1}^2 \leq 2\left[u_n^4 + \exp(-2^{n+1} c \log t/a)\right],$$

whence

(17) $$u_{n+2} \leq u_{n+1}^2 + \exp(-2^{n+1} c \log t/a) \leq 2u_n^4 + 3 \exp(-2^{n+1} c \log t/a).$$

Repeating the process, we obtain at the k-th step

(18) $$u_{n+k} \leq v_k u_n^{2^k} + w_k \exp(-2^{n+k-1} c \log t/a)$$

where v_k and w_k are constants for which we will now obtain upper bounds. In fact we have

(19) $$u_{n+k} \leq 2^{2^{k-1}-1} u_n^{2^k} + 3^{2^{k-1}-1} \exp\left(-2^{n+k-1} c \log t/a\right),$$

the validity of (19) being readily established by induction. Hence,

taking $n = 0$,

(20) $$\Psi\left(t^{2^k}\right) \leq 2^{2^{k-1}-1}[\Psi(t)]^{2^k} + 3^{2^{k-1}-1}\exp(-2^{k-1}c\log t/a).$$

It follows from Lemma 1 that we may choose $t_0 > 1$ and large enough so that $\Psi^2(t_0) < \frac{1}{2}$ and $c \log t_0/a > \log 3$. Then if $x \geq t_0^2$ we may write

(21) $$x = t^{2^k}, \qquad t_0 \leq t < t_0^2,$$

where (21) defines uniquely the positive integer k. Then (20) gives

(22) $$\Psi(x) \leq \frac{1}{2}[2\Psi^2(t)]^{2^{k-1}} + \frac{1}{3}[\exp(\log 3 - c\log t/a)]^{2^{k-1}}$$

$$< \frac{1}{2}[2\Psi^2(t_0)]^{\frac{1}{2}L} + \frac{1}{3}[\exp(\log 3 - c\log t_0/a)]^{\frac{1}{2}L},$$

where L is the logarithm of x to the base t_0^2. From (22), Lemma 2 follows immediately for $t \longrightarrow \infty$; similarly for $t \longrightarrow -\infty$.

<u>Lemma</u> 3. <u>If</u> $|\phi(it)| = O(|t|^{-d})$, $d > 0$, <u>as</u> $t \longrightarrow \pm\infty$, <u>then</u> $|\phi(it)| = O(|t|^{-c/a})$.

<u>Proof</u>: The relation $|\phi(it)| = O(|t|^{-d})$ may be written $|\phi(it)| = O[(1 + |t|)^{-d}]$, and in this form it is more convenient for our purposes. We have

(23) $$|\phi(it)| \leq c_1^2 \int_0^\infty (1 + |t|e^{-ay})^{-2d} dG(y).$$

Integrating by parts, this becomes

$$(24)\qquad \leq \left[-c_1^2 \int_y^\infty dG(y)/(1 + |t|e^{-ay})^{2d}\right]_0^\infty$$

$$+ c_1^2 \int_0^\infty \left[\int_y^\infty dG(x)\right]\left[\frac{d}{dy}(1 + |t|e^{-ay})^{-2d}\right]dy$$

$$\leq \frac{c_1^2}{(1 + |t|)^{2d}} + c_2 \int_0^\infty e^{-cy}\left[\frac{d}{dy}(1 + |t|e^{-ay})^{-2d}\right]dy .$$

Integrating by parts again, we obtain

$$(25)\qquad \leq \frac{c_3}{(1 + |t|)^{2d}} + c_4 \int_0^\infty \frac{e^{-cy}\,dy}{(1 + |t|e^{-ay})^{2d}} .$$

Make the substitution $e^{ay} = |t|v$, obtaining

$$(26)\qquad \leq \frac{c_3}{(1 + |t|)^{2d}} + c_5 \left(\int_{1/|t|}^\infty \frac{v^{-c/a-1}\,dv}{(1 + 1/v)^{2d}}\right)|t|^{-c/a} .$$

The integral is

$$(27)\qquad \int_{1/|t|}^\infty \frac{v^{-c/a-1+2d}}{(v + 1)^{2d}}dv = O\left[\max(1, |t|^{c/a-2d})\right], \qquad c/a \neq 2d,$$

as $t \longrightarrow \infty$.

Hence if $2d > c/a$, we obtain the desired estimate. If not, we obtain $|\phi(it)| = O(|t|^{-2d})$. This process may now be repeated using the new estimate, and since $2^n d > c/a$ for some integer n, we will eventually obtain the desired result. Clearly d may be picked so that $2^n d$ is never equal to c/a.

This completes the proof of Theorem 6.

It is possible to continue in this way and obtain bounds for $\phi'(it)$. However, since we are principally interested in showing that

$$(28) \qquad \int_{-\infty}^{\infty} |\phi'(it)|dt < \infty,$$

we shall merely show that this is implied by the relation $|\phi(it)| = O(t^{-d})$, $d > 0$, and by the integral equation satisfied by ϕ.

We have

$$(29) \qquad \phi'(s) = 2\int_0^{\infty} \phi'(se^{-ay})\phi(se^{-ay})e^{-ay}\, dG(y).$$

From this we obtain for $T \geq 0$,

$$(30) \qquad \int_0^T |\phi'(it)|dt \leq 2\int_0^{\infty} \left\{\int_0^T |\phi'(ite^{-ay})|\ |\phi(ite^{-ay})|dt\right\} e^{-ay}dG(y).$$

In the inner integral make the substitution $te^{-ay} = u$. Then

$$(31) \qquad B(T) = \int_0^T |\phi'(it)|dt \leq 2\int_0^{\infty}\left\{\int_0^{Te^{-ay}} |\phi'(it)|\ |\phi(it)|dt\right\}dG(y)$$

$$\leq 2\int_0^T |\phi'(it)|\ |\phi(it)|dt.$$

Since $\phi(it) = O(t^{-d})$ as $t \longrightarrow \infty$, we have for some constant c,

$$(32) \qquad B(T) \leq c\int_0^T |\phi'(it)|dt/(1 + t)^d.$$

Integrating now by parts,

$$(33) \qquad B(T) \leq cB(T)/(1 + T)^d + cd\int_0^T B(t)dt/(1 + t)^{d+1}.$$

Choose T large enough so that $c/(1 + T)^d \leq 1/2$; then

(34) $$B(T) \leq 2cd \int_0^T B(t)dt/(1 + t)^{d+1} .$$

Hence

(35) $$B(T)/(1 + T)^{d+1} \leq \left[2cd/(1 + T)^{d+1}\right] \int_0^T B(t)dt/(1 + t)^{d+1}$$

or, setting $V(T) = \int_0^T B(t)dt/(1 + t)^{d+1}$,

(36) $$V'(T) \leq 2cdV(T)/(1 + T)^{d+1} .$$

Hence $V(T)$, and thus $B(T)$, are bounded as $T \longrightarrow \infty$. A similar argument holds as $t \longrightarrow -\infty$.

§9. Existence of a Density Function.

From the results of the previous section follows

Theorem 10: *The distribution* $K(u)$ *of* w *is a continuous function of* u. *If, in addition,* $1 - G(y) = O(e^{-cy})$, $c > 0$, *then* $K(u)$ *is absolutely continuous,* $K(u) = \int_0^u k(v)dv$.

Part 1 of the theorem follows from the fact that $\phi(it) \longrightarrow 0$ as $t \longrightarrow \infty$; part 2 follows, using (1) of §8, and the fact that $\int_{-\infty}^{\infty} |\phi'(it)|dt < \infty$, cf. the argument in [3], p. 480.

§10. An Example.

Suppose

$$G(t) = \frac{b^n}{\Gamma(n)} \int_0^t y^{n-1} e^{-by} dy.$$

Let

$$A(s) = \int_0^\infty e^{-st} dG(t) = (1 + s/b)^{-n}.$$

Proper choice of n and b give any desired values for the mean life length, $-A'(0) = n/b$, and the variance, $A''(0) - [A'(0)]^2 = n/b^2$. The root of $A(s) = \frac{1}{2}$ is given by

$$a = b(2^{1/n} - 1),$$

and

$$E[Z(t)] \sim (a + b)e^{at} / (2an).$$

Similarly,

$$E[Z(t)]^2 - [EZ(t)]^2 \sim (4I_2 - 1)e^{2at}(a + b)^2 / [4a^2n^2(1 - 2I_2)]$$

where $I_2 = A(2a) = (1 + 2a/b)^{-n}$.

§11. Remarks.

The methods already employed can be used to treat the case where instead of binary fission there is a probability q_n, $n = 0, 1, 2, \cdots$, of transformation into n particles. The proofs for existence, uniqueness, and generating-function properties of

$F(s, t)$ are essentially unmodified. The rest of the treatment depends on the value of $m = \sum_{n=0}^{\infty} nq_n$. Just as in the simpler case treated in [3] there is a positive probability that $Z(t)$ never vanishes if and only if $m > 1$. If $m > 1$ and $\sum n^k q_n < \infty$, the results on the asymptotic behavior of $m_j(t)$, $j \leq k$, are the same except for different values of the constants involved; the theorems on mean square convergence of $Z(t)/EZ(t)$ hold if $m > 1$ and $\sum n^2 q_n < \infty$. The theorem on analyticity of $\phi(s)$ at $s = 0$ is not generally true but presumably holds if the radius of convergence of $h(s) = \sum q_n s^n$ is greater than 1. The function $\phi(s)$ satisfies

(1) $$\phi(s) = \int_0^\infty h[\phi(se^{-ay})]dG(y),$$

$$1 = m \int_0^\infty e^{-at} dG(t).$$

From (1) we see that $\phi(-\infty) = h[\phi(-\infty)] = Q$, so that Q, the probability that $w = 0$ (and also the probability that for some t, $Z(t) = 0$), is the unique nonnegative root, less than 1, of

(2) $$Q = h(Q).$$

The transformations

$$h^*(s) = \frac{h[s(1 - Q) + Q] - Q}{1 - Q}$$

$$\phi^*(s) = \frac{\phi[(1 - Q)s] - Q}{1 - Q}$$

make (1) take the form

$$(3) \qquad \phi^*(s) = \int_0^\infty h^*[\phi^*(se^{-ay})]dG(y).$$

From (3), using the methods of Section 8, we can show

$$\int_{-\infty}^{\infty} |\phi^{*\prime}(it)|dt < \infty,$$

showing that the distribution of w is absolutely continuous except for a jump of magnitude Q at 0.

The condition $G(0+) = 0$ can be partly dispensed with. The condition $G(0+) > 0$ means that an instantaneous chain reaction may occur at the very instant of birth of each particle, producing a whole family at once. If $h'(1)G(0+) \leq 1$ it can be shown that the number of particles produced in a finite length of time is finite with probability 1, and a treatment analogous to that of this paper can be given.

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